ELEMENTARY CONSEQUENCES OF THE NONCONTRACTIBILITY OF THE CIRCLE

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Most topology texts prove the noncontractibility of the circle as an easy corollary of the computation of its fundamental group or its homology groups. Thus, this property of the circle has become part of a relatively advanced subject: algebraic topology.

The noncontractibility of the circle can be proved quite easily using only material from a traditional undergraduate point-set topology course. Furthermore, a large number of interesting and significant consequences of this fact can be obtained without additional mathematical prerequisites. Therefore, a fascinating area of elementary topology has been inaccessible to a substantial number of people who do, in fact, have sufficient background in topology to understand and appreciate it. The purpose of this paper is to try to correct this oversight.

1. Noncontractibility of the circle. Maps f, g: $X \to Y$ are *homotopic* if there is a map (*homotopy*) $h : X \times I \to Y$, where $I = [0,1]$, such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$ for all x in X. A space X is *contractible* if the identity map on X is homotopic to a constant map. Such a homotopy is called a *contraction* of X.

Denote the circle of radius one with center at the origin of the plane by S^1 and the real numbers by R.

The following result has been attributed to Eilenberg [**2**, p. 361]. It is also proved as Theorem 1 in [**4**].

THEOREM. *If $f : S^1 \to S^1$ is a map homotopic to the constant map, then there exists a map $\phi : S^1 \to R$ such that $f(x) = e^{i\phi(x)}$ for all $x \in S^1$.*

Proof. Suppose that $g : S^1 \to S^1$ is a map for which there is a map $\phi : S^1 \to R$ with $g(x) = e^{i\phi(x)}$for all $x \in S^1$. We claim that if $h : S^1 \to S^1$ is a map such that $|g(x) - h(x)| < 2$ for all $x \in S^1$, then $f(x) = e^{i\psi(x)}$ for some map $\psi : S^1 \to R$. To prove the claim, note that $|g(x) - h(x)| < 2$ implies $h(x) \neq -g(x)$ so $h(x)/g(x) \neq -1$. Let $\lambda(x)$ be the number of radians in the angle between 1 and $h(x)/g(x)$ if $h(x)/g(x)$ is above the x-axis, and the negative of that number if $h(x)/g(x)$ is below. Then $h(x)/g(x) = e^{i\lambda(x)}$, or

$$h(x) = g(x)e^{i\lambda(x)} = e^{i(\phi(x)+\lambda(x))} = e^{i\psi(x)},$$

which verifies the claim. Now let $H : S^1 \times I \to S^1$ be a homotopy such that $H(x,0) = x_0$ (constant) and $H(x,1) = f(x)$. By uniform continuity, there exists $\delta > 0$ such that $|H(x,t) - H(x,t')| < 2$, for $|t - t'| < \delta$ and all $x \in S^1$. Partition $[0,1]$ by $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = 1$ so that $|t_{i+1} - t_i| < \delta$. A constant map from S^1 to itself is of the form $e^{i\phi(x)}$ where $\phi : S^1 \to R$ is constant, so $H(x, t_1) = e^{i\phi_1(x)}$ for some $\phi_1 : S^1 \to R$. Repeating the argument $n - 1$ times completes the proof.

THEOREM. *The circle is not contractible.* (Note [**2**, p. 364].)

Proof. Assuming the contrary, the previous theorem implies the existence of a map $\phi : S^1 \to R$ such that $x = e^{i\phi(x)}$ for all $x \in S^1$. The map ϕ is one-to-one and so the function $g : S^1 \to \{-1, 1\}$ given by

$$g(x) = \frac{\phi(x) - \phi(-x)}{|\phi(x) - \phi(-x)|}$$

is well-defined and continuous. But $g(-x) = -g(x)$, so g takes the connected space S^1onto a disconnected space; which is impossible.

2. Elementary consequences. Let R^2 denote the plane, let D^2 be the disc of radius one centered at the origin, and consider S^1 as the boundary of D^2.

A. (Knaster, Kuratowski and Mazurkiewicz [7]). *Given a map* $f : D^2 \to R^2$ *such that* $f(S^1) \subseteq D^2$, *there exists* $x_0 \in D^2$ *with* $f(x_0) = x_0$.

Proof. (Benjamin Halpern) Define $r : R^2 - 0 \to S^1$ by $r(x) = x/|x|$. If $f(x) \neq x$ for all $x \in D^2$, then $H : S^1 \times I \to S^1$ defined by

$$H(x,t) = \begin{cases} r[x - 2tf(x)] & \text{if } 0 \leqq t \leqq \frac{1}{2} \\ r[(2-2t)x - f((2-2t)x)] & \text{if } \frac{1}{2} \leqq t \leqq 1 \end{cases}$$

would be a contraction of S^1, contrary to what has been established. (Note that if $t < \frac{1}{2}$ then $2t|f(x)| < |x|$ so H is well-defined.)

B. (Brouwer Fixed Point Theorem). *Given a map* $f : D^2 \to D^2$, *there exists* $x_0 \in D^2$ *with* $f(x_0) = x_0$.

C. *Let* $I^2 = I \times I$ *and define* $B_1 = \{1\} \times I$, $B_1' = \{0\} \times I$, $B_2 = I \times \{1\}$, $B_2' = I \times \{0\}$. *Suppose that* A_1 *and* A_2 *are closed subsets of* I^2 *such that, for* $i = 1, 2$, B_i *and* B_i' *are in different components of* $I^2 - A_i$, *then* $A_1 \cap A_2 \neq \emptyset$.

There is an elementary proof of this statement on pages 40–41 of [**6**] — as the case $n = 2$ of "Proposition D." We shall not repeat the argument here.

If $A \subseteq X$ and $f : X \to A$ is a map such that $f(a) = a$ for all $a \in A$, then f is a *retraction* of X onto A.

D. *There is no retraction of* D^2 *onto* S^1.

Proof. Suppose a retraction $f : D^2 \to S^1$ exists. Let $g : S^1 \to S^1$ be a nontrivial rotation, then $gf : D^2 \to S^1 \subseteq D^2$ contradicts the Brouwer theorem.

Let R^3 denote three-dimensional euclidean space. Two circles J and K in R^3 are *unlinked* if there exists a map $f : D^2 \to R^3 - K$, such that the restriction of f to S^1 is a homeomorphism onto J. Otherwise J and K are *linked.*

E. *Let* J *be the circle of radius one in the plane* $y = 0$ *with center* $(1,0,0)$, *and* *let K be the circle of radius one in the plane $z = 0$ with center* $(0,0,0)$. *Then J and K are linked.*

Proof. Let A be the half-plane of points $(x,0,z)$ such that $x \geqq 0$. Then radial projection σ from $(1,0,0,)$ retracts $A - (1,0,0)$ onto J. The map $g : R^3 \to R^3$ defined by

$$g(x,y,z) = ((x^2 + y^2)^{\frac{1}{2}}, 0, z)$$

retracts $R^3 - K$ onto $A - (1,0,0)$. Suppose J and K are unlinked, and let $f : D^2 \to R^3 - K$ be a map such that $h : S^1 \to J$, the restriction of f, is a homeomorphism. The retraction

$$h^{-1}\sigma g f : D^2 \to S^1$$

contradicts the previous result.

The noncontractibility of the circle and Theorems A–E are all closely related and can be proved one from another without difficulty. (Compare the remark on page 41 of [**6**].)

F. *If $f : D^2 \to R^2$ is a map such that for $x \in S^1$, either $f(x) = (0,0)$, or x does not lie on the ray from the origin through $f(x)$, then $f(x_0) = x_0$ for some $x_0 \in D^2$* (see [**2**, p. 353]).

Proof. Define $g : D^2 \to D^2$ by

$$g(x) = \begin{cases} f(x) & \text{if } f(x) \in D^2 \\ f(x)/|f(x)| & \text{otherwise.} \end{cases}$$

Then $g(x_0) = x_0$ for some $x_0 \in D^2$ by the Brouwer theorem. If $f(x_0) \notin D^2$, then $x_0 = g(x_0) \in S^1$. But $g(x_0)$ does lie on the ray from the origin through $f(x_0)$, so $f(x_0)$ must be in D^2 and therefore $f(x_0) = x_0$.

G. (Fundamental Theorem of Algebra). *Every nonconstant polynomial with complex coefficients has a complex root.*

An elementary proof, based on the Brouwer theorem, can be found in [**4**, Theorem 6].

Given a map $f : X \to Y$, *the mapping cylinder $M(f)$* of the map is the quotient space of the disjoint union $(X \times I) \cup Y$ under the equivalence relation: $(s,1) \sim f(x)$ for all $x \in X$. Note that there is an embedding $i : Y \to M(f)$ taking $y \in Y$ to its equivalence class $[y]$.

H. *The circle is not a mapping cylinder*; *that is, there are no spaces X, Y and map $f : X \to Y$ such that $M(f)$ is homeomorphic to S^1* (see [**5**, p. 159]).

Proof. Suppose there is a homeomorphism $h : M(f) \to S^1$ for some map $f : X \to Y$.

Let $x_0 \in X$, then $h(x_0, 0) \notin hi(Y)$. Define $H: M(f) \times I \to M(f)$ by

$$H([y], s) = [y] \qquad \text{for } y \in Y,$$
$$H([x, t], s) = [x, (1-s)t + s] \qquad \text{for } (x, t) \in X \times I.$$

Then $H([x, t], 0) = [x, t]$. Define $H_1: M(f) \to i(Y)$ by setting $H_1([y]) = [y]$ and $H_1([x, t]) = [x, 1]$. Then H_1 is a retraction. The existence of the homotopy H proves that the identity map on S^1 is homotopic to the composition:

$$S^1 \xrightarrow{h^{-1}} M(f) \xrightarrow{H_1} i(Y) \xrightarrow{h} hi(Y) \subseteq (S^1 - h(x_0, 0)) \subseteq S^1.$$

On the other hand, $S^1 - h(x_0, 0)$ is homeomorphic to R and therefore contractible, so the composition is homotopic to a constant map. Thus we obtain a contraction of S^1, contrary to the first section of this paper.

Let C denote the complex numbers topologized by identifying C with R^2. Let $C^{n+1} - 0$ be the space of all $(n+1)$-tuples $(z_0, z_1, \cdots, z_n)$ of complex numbers such that $\sum_{j=0}^{n} z_j \bar{z}_j \neq 0$. *Complex projective n-space* CP^n is the quotient space of $C^{n+1} - 0$ under the equivalence relation $\sim$, where $(z_0, z_1, \cdots, z_n) \sim (z'_0, z'_1, \cdots, z'_n)$ if and only if

$$(z'_0, z'_1, \cdots, z'_n) = \lambda(z_0, z_1, \cdots, z_n) = (\lambda z_0, \lambda z_1, \cdots, \lambda z_n)$$

for some $\lambda \in C$. Denote the equivalence class containing $(z_0, z_1, \cdots, z_n)$ by $[z_0, z_1, \cdots, z_n]$.

I. *It is not possible to choose representatives of the elements of* CP^n *in a continuous manner; that is, there is no map* $\sigma: CP^n \to C^{n+1} - 0$ *such that* $\sigma[z_0, z_1, \cdots, z_n]$ *is a member of* $[z_0, z_1, \cdots, z_n]$.

Proof. Let $p: C^{n+1} - 0 \to CP^n$ be the quotient map. The identification of C with R^2 makes $S^1 = \{z \in C \mid z\bar{z} = 1\}$. Assume that σ exists and define $f: S^1 \to C^{n+1} - 0$ by $f(z) = z^{-1}\sigma[z, z, \cdots, z]$. For $(z_0, z_1, \cdots, z_n)$ in $C^{n+1} - 0$, we know that $\sigma[z_0, z_1, \cdots, z_n] = \lambda(z_0, z_1, \cdots, z_n)$ for some nonzero complex number λ. Define $g: C^{n+1} - 0 \to S^1$ by $g(z_0, z_1, \cdots, z_n) = \lambda / |\lambda|$. Now, for $z \in S^1$,

$$\begin{aligned} \sigma p(f(z)) &= \sigma p(z^{-1}\sigma[z, z, \cdots, z]) \\ &= \sigma p \sigma[z, z, \cdots, z] \\ &= \sigma[z, z, \cdots, z] = zf(z). \end{aligned}$$

Thus, since $|z| = 1$, we have $gf(z) = z$ for all $z \in S^1$. Let $\Delta = \{(z, z, \cdots, z) \in C^{n+1} - 0\}$ and define $H: \Delta \times I \to C^{n+1} - 0$ by

$$H((z, z, \cdots, z), t) = \begin{cases} ((1-2t)z + 2t, z, \cdots, z) & \text{if } 0 \leqq t \leqq \frac{1}{2} \\ (1, (2-2t)z, \cdots, (2-2t)z) & \text{if } \frac{1}{2} \leqq t \leqq 1. \end{cases}$$

Then H is a homotopy between the inclusion map of Δ in $C^{n+1} - 0$ and a constant map. Observe that $f(S^1) \subseteq \Delta$ and define $G : S^1 \times I \to S^1$ by $G(z,t) = gH(f(z),t)$. Then $G(z,0) = gf(z) = z$ while $G(z,1) = g(1,0,\cdots,0)$, so that G would be a contraction of S^1. But this is impossible.

J. (Jordan Curve Theorem) *If S is a subset of R^2 homeomorphic to the circle, then $R^2 - S$ has precisely two components.*

An elementary proof can be found in [**2**, p. 362].

3. Generalizations. It is possible to continue the program of this paper and prove the noncontractibility of all n-spheres S^n without explicitly using algebraic topology. For example, there are proofs, based on the theory of the "degree modulo two" of a map of spheres, in [**6**, pp. 37–40] and [**8**, pp. 20–25]. Another alternative is to begin either with the combinatorial proof of the Brouwer Fixed Point Theorem in n-dimensions based on Sperner's Lemma ([**1**, pp. 155–169] or [**11**]), with Tucker's proof [**12**] of Brouwer's Theorem, or with Milnor's proof [**8**, p. 24] of the same theorem. Then the Brouwer theorem obviously implies that there is no retraction of D^n onto S^{n-1} (compare Theorem D of Section 2). Suppose that S^n were contractible and let $H : S^n \times I \to S^n$ be a map where $H(x,0) = x_0$ and $H(x,1) = x$, for all $x \in S^n$. Define $g : D^n \to S^n \times I$ by letting $g(x) = (x/|x|, |x|)$ if x is not the origin and sending the origin to $(x_0, 0)$. Then g is not continuous, but Hg is continuous — in fact, Hg is a retraction of D^n onto S^{n-1}. We conclude that S^n is not contractible.

Assuming that S^n is not contractible, the statements and proofs of Theorems A, B, C, D, F and H of Section 2 generalize to arbitrary dimensions with only trivial changes. Theorem E generalizes to linked n-spheres in R^{2n+1}; with obvious modifications of the proof. There is a form of the Fundamental Theorem of Algebra (Theorem G) for polynomials with quaternion coefficients or with Cayley number coefficients, but its proof requires some algebraic topology [**3**, p. 308]. Theorem I is essentially the fact that the Hopf fibering of S^3 over S^2 and, more generally, the fibre bundle S^{2n+1} over CP^n, admits no cross-section. Assuming S^3 noncontractible, the same proof will show that the fibre bundle S^{4n+3} over quaternionic projective n-space, in particular the Hopf fibering of S^7 over S^4, has the same property (see [**10**, pp. 106–108]). Theorem J extends to all dimensions (the Jordan-Brouwer Separation Theorem [**9**, p. 198]), but I know of no elementary proof. However, the geometric proof of a weaker statement — that if $S \subseteq R^2$ is homeomorphic to S^1 then $R^2 - S$ is disconnected — does generalize [**2**, p. 358].

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References

1. P. Aleksandrov, Combinatorial topology Vol. 1, Graylock Press, New York, 1956.
2. J. Dugundji, Topology, Allyn and Bacon, Boston, 1966.

3. S. Eilenberg and N. Steenrod, Foundations of algebraic topology, Princeton, N. J., 1952.
4. M. Fort, Some properties of continuous functions, this MONTHLY, 59 (1952) 372–375.
5. J. Hocking and G. Young, Topology, Addison-Wesley, Reading, Mass., 1961.
6. W. Hurewicz and H. Wallman, Dimension theory, Princeton, N. J., 1941.
7. B. Knaster, K. Kuratowski and S. Mazurkiewicz, Ein Beweis des Fixpunktsatzes für n-dimensionale Simplexe, Fund. Math., 14 (1929) 132–137.
8. J. Milnor, Topology from a differentiable viewpoint, Univ. of Virginia, 1965.
9. E. Spanier, Algebraic topology, McGraw-Hill, New York, 1966.
10. N. Steenrod, The topology of fibre bundles, Princeton, N. J., 1951.
11. C. Tompkins, Sperner's lemma and some extensions, Applied Combinatorial Math. (E. Beckenbach, Ed.), Wiley, New York, 1964, 416–455.
12. A. Tucker, Some topological properties of disk and sphere, Proc. First Canadian Math. Congress, 1945, 285–309.

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THE SECOND U.S.A. MATHEMATICAL OLYMPIAD

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The U.S.A. Mathematical Olympiad is a new venture, whose purpose is to attempt to discover secondary school students with superior mathematical talent — who possess creativity and inventiveness as well as computational skills. Participation is limited to about 100 students selected mainly from the Honor Roll of the Annual High School Mathematics Examination, plus a few recommended students from those states which sponsor their own High School mathematics competitions. The Olympiad consists of five essay-type problems, requiring mathematical power on the part of the participants.

1. Introduction. In 1973, 123 invitations were sent out, and 107 completed acceptances were received. (An acceptance is "complete" when the student agrees to participate *and* the school agrees to administer the test.) The Second U.S.A. Mathematical Olympiad took place on May 1, 1973. It is reproduced below.

SECOND U.S.A. MATHEMATICAL OLYMPIAD — MAY 1, 1973

1. Two points, P and Q, lie in the interior of a regular tetrahedron $ABCD$. Prove that angle $PAQ < 60°$.

2. Let $\{X_n\}$ and $\{Y_n\}$ denote two sequences of integers defined as follows:

$$X_0 = 1, X_1 = 1, \quad X_{n+1} = X_n + 2X_{n-1} \quad (n = 1, 2, 3, \cdots),$$

$$Y_0 = 1, \ Y_1 = 7, \quad Y_{n+1} = 2Y_n + 3Y_{n-1} \quad (n = 1, 2, 3, \cdots).$$